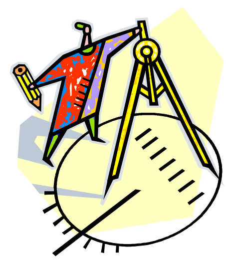
THE CIRCLE – CENTER AT THE ORIGIN

We're now ready for a new type of graph. In this chapter, we analyze "nature's perfect shape," the circle. Whereas the equation of a line has no squared variables, and a parabola has one squared variable, we will see that the equation of a circle has both variables squared.

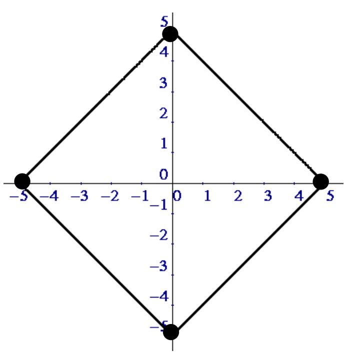


❑ *GRAPHING A CIRCLE*

EXAMPLE 1: **Graph:** $\boldsymbol{x^2 + y^2 = 25}$

Solution: The graph is undoubtedly not a line, since the variables are squared. Let's plot points and see what we get.

Let's first check out the **intercepts**. If we set $x = 0$, the resulting equation is $y^2 = 25$, whose two solutions are $y = \pm 5$. Thus, there are two y-intercepts, **(0, 5)** and **(0, –5)**. You can set y to 0 and calculate the x-intercepts to be **(5, 0)** and **(–5, 0)**. We now have four points on our graph, but it's unclear how to connect them — maybe the graph looks like a diamond? We'll find some other points; for example, if we let $x = 3$, then



Are four points enough to make an accurate graph? Perhaps, perhaps not.

$$3^2 + y^2 = 25 \Rightarrow 9 + y^2 = 25$$

$$\Rightarrow y^2 = 16 \Rightarrow y = \pm 4$$

$\Rightarrow$ **(3, 4)** and **(3, –4)** are on the graph.

If x is chosen to be -3, then

$$(-3)^2 + y^2 = 25 \Rightarrow 9 + y^2 = 25 \Rightarrow y^2 = 16 \Rightarrow y = \pm 4$$

$\Rightarrow$ **(–3, 4)** and **(–3, –4)** are also on the graph.

Now let $x = 4$:

$$4^2 + y^2 = 25 \Rightarrow 16 + y^2 = 25 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$$

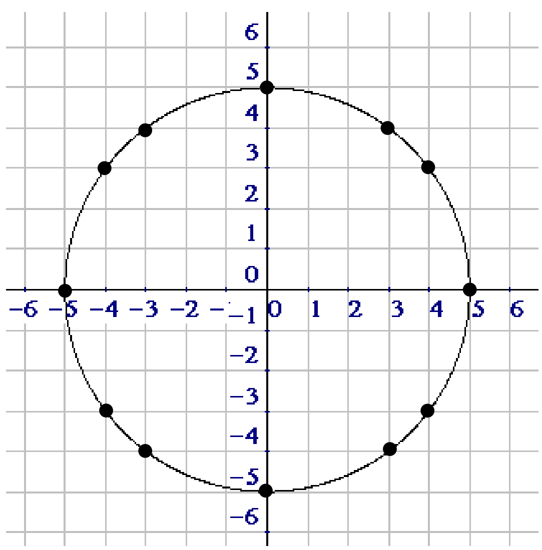
$\Rightarrow$ **(4, 3)** and **(4, –3)** are on the graph.

Our last choice for x will be -4:

$$(-4)^2 + y^2 = 25 \Rightarrow 16 + y^2 = 25 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$$

$\Rightarrow$ **(–4, 3)** and **(–4, –3)** are also on the graph.

This set of 12 points should be enough data to get a decent picture, which looks a lot like a circle:



<u>Summary:</u> The graph of

$$x^2 + y^2 = 25$$

is a circle with its center at the origin, (0, 0), and with a radius of 5.

Homework

1. For each circle, determine the four **intercepts**:

a. $x^2 + y^2 = 1$ b. $x^2 + y^2 = 2$ c. $x^2 + y^2 = 4$

d. $x^2 + y^2 = 7$ e. $x^2 + y^2 = 49$ f. $x^2 + y^2 = 60$

2. Given the circle and the *x*-value (i.e., an input), find the *y*-values (i.e., the outputs):

a. $x^2 + y^2 = 100;\ x = 6$ b. $x^2 + y^2 = 100;\ x = -8$

c. $x^2 + y^2 = 1;\ x = 1$ d. $x^2 + y^2 = 169;\ x = 5$

e. $x^2 + y^2 = 169;\ x = -12$ f. $x^2 + y^2 = 169;\ x = -5$

g. $x^2 + y^2 = 10;\ x = 2$ h. $x^2 + y^2 = 12;\ x = -2$

3. Consider again the circle $x^2 + y^2 = 25$ from Example 1. Its center is the origin and its radius is 5. Now look at the graph and notice that there's no point on the graph with an x-value of 8. Prove this fact algebraically (that is, use the equation).

EXAMPLE 2:

A. Consider the circle $x^2 + y^2 = 81$. Using Example 1 as a guide, we infer that its center is the **origin** and its radius is **9**.

B. Now look at the circle $x^2 + y^2 = 13$. The center is **(0, 0)**, and the radius is $\boldsymbol{\sqrt{13}}$.

C. If the center of a circle is the origin, and if its radius is 15, what is the equation of the circle? It's $\boldsymbol{x^2 + y^2 = 225}$.

D. What is the equation of the circle with center (0, 0) and radius $3\sqrt{7}$?

$$x^2 + y^2 = \left(3\sqrt{7}\right)^2, \text{ which is } \boldsymbol{x^2 + y^2 = 63}.$$

Note: $\left(3\sqrt{7}\right)^2 = 3^2 \cdot \sqrt{7}^2 = 9 \cdot 7 = 63$

Homework

4. Find the center and radius of each circle:

a. $x^2 + y^2 = 25$
b. $x^2 + y^2 = 144$
c. $x^2 + y^2 = 1$
d. $x^2 + y^2 = 17$
e. $x^2 + y^2 = 27$
f. $x^2 + y^2 = 200$
g. $x^2 + y^2 = 0$
h. $x^2 + y^2 = -9$
i. $x + y = 10$
j. $x^2 + y = 49$

Note: Working with the radius can be confusing. If given the circle equation in standard form, the radius of the circle is found by taking the positive square root of the number to the right of the equal sign. On the other hand, if you know the radius, you square it when you put it into the formula.

5. Find the equation of the circle with center at the origin and the given radius:

a. $r = 10$
b. $r = 25$
c. $r = 1$
d. $r = 16$
e. $r = \sqrt{11}$
f. $r = \sqrt{18}$
g. $r = 4\sqrt{5}$
h. $r = 3\sqrt{7}$

6. The ***unit circle*** is the circle whose *center* is at the origin and whose *radius* is 1.

a. Where is the center of the unit circle?
b. What is the radius of the unit circle?
c. What is the equation of the unit circle?
d. What is the area of the unit circle?
e. What is the circumference of the unit circle?

7. Consider the circle $x^2 + y^2 = 7$.
   a. What is the radius?
   b. What is the diameter?
   c. What is the circumference?
   d. What is the area?

8. Sketch a circle (with center not necessarily at the origin) that has exactly
   a. 4 intercepts
   b. 2 intercepts
   c. 3 intercepts
   d. 1 intercept
   e. 0 intercepts

9. Consider the equation $x^2 + y^2 = k$. Describe the graph of this equation for each situation:

   a. $k > 0$ b. $k = 0$ c. $k < 0$

Review Problems

10. Find the center and radius of the circle $x^2 + y^2 = 20$.

11. True/False:

   a. Every circle has at least one intercept.
   b. The radius of the circle $x^2 + y^2 = 1$ is 1.
   d. $x^2 + y^2 + 4 = 3$ is a circle.
   e. $x^2 + y^2 = 1$ is called the *unit circle*.
   f. The area of the circle $x^2 + y^2 = 25$ is 25π.

Solutions

1. a. $(\pm1, 0)$ $(0, \pm1)$ b. $(\pm\sqrt{2}, 0)$ $(0, \pm\sqrt{2})$

c. $(\pm2, 0)$ $(0, \pm2)$ d. $(\pm\sqrt{7}, 0)$ $(0, \pm\sqrt{7})$

e. $(\pm7, 0)$ $(0, \pm7)$ f. $(\pm2\sqrt{15}, 0)$ $(0, \pm2\sqrt{15})$

2. a. ±8 b. ±6 c. 0 d. ±12

e. ±5 f. ±12 g. $\pm\sqrt{6}$ h. $\pm2\sqrt{2}$

3. Hint: Let $x = 8$ in the circle equation and try to solve for y.

4. a. $C(0, 0)$ $r = 5$ b. $C(0, 0)$ $r = 12$ c. $C(0, 0)$ $r = 1$

d. $C(0, 0)$ $r = \sqrt{17}$ e. $C(0, 0)$ $r = 3\sqrt{3}$ f. $C(0, 0)$ $r = 10\sqrt{2}$

g. It's not a circle; the graph is just the origin.

h. It's not a circle; there are two reasons. First, the radius would be $\sqrt{-9}$, which is not a real number. Second, if two numbers are squared and then added together, there's no way that sum could be negative.

i. It's a line, not a circle.

j. It's not a circle — do you know what it is?

5. a. $x^2 + y^2 = 100$ b. $x^2 + y^2 = 625$ c. $x^2 + y^2 = 1$

d. $x^2 + y^2 = 256$ e. $x^2 + y^2 = 11$ f. $x^2 + y^2 = 18$

g. $x^2 + y^2 = 80$ h. $x^2 + y^2 = 63$

6. a. $(0, 0)$, the origin b. 1 c. $x^2 + y^2 = 1$

d. $A = \pi r^2 = \pi(1^2) = \pi(1) = \boldsymbol{\pi}$ e. $C = 2\pi r = 2\pi(1) = \boldsymbol{2\pi}$

7. a. $r = \sqrt{7}$ b. $d = 2\sqrt{7}$ c. $C = 2\sqrt{7}\,\pi$ d. $A = 7\pi$

8. You're on your own.

9. a. If $k > 0$, the graph is a circle with center at the origin and radius $\sqrt{k}$.

b. If $k = 0$, the graph is just the single point (0, 0); i.e., the origin.

c. If $k < 0$, the graph is empty (there's no graph at all).

10. C(0, 0); $r = 2\sqrt{5}$

11. a. F b. T d. F e. T f. T

"Only those who dare to fail greatly can ever achieve greatly."

– Robert F. Kennedy